Complete collineations revisited

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0 Introduction

This paper takes a new look at some old spaces.

The old spaces are the *moduli spaces of complete collineations*, introduced and explored by many of the leading lights of 19th-century algebraic geometry, such as Chasles, Schubert, Hirst, and Giambelli. They are roughly compactifications of the spaces of linear maps of a fixed rank between two fixed vector spaces, in which the boundary added is a divisor with normal crossings. This renders them useful in solving many enumerative problems on linear maps, and they are famous as much for the intricacy of the resulting formulas as for the elegance and symmetry of the underlying geometry.

The new look comes from some recent quotient constructions in algebraic geometry: the Chow quotient and inverse limit of Mumford quotients, introduced in 1991 by Kapranov, Sturmfels, and Zelevinsky, and the Hilbert quotient, essentially due to Bialynicki-Birula and Sommese in 1987. These were motivated partly by the search for a quotient more canonical than the Mumford or geometric invariant theory quotient, which depends on the choice of a linearization, and partly by the attractive polyhedral interpretations possessed by all three quotients in the setting of toric geometry. But for us, their chief interest arose later, in two papers of Kapranov from 1993. They showed, among many other things, that Chow quotients, and several related operations, could be used to give elegant new constructions of the moduli space $\overline{M}_{0,n}$ of stable punctured curves of genus zero.

This paper will demonstrate that every one of Kapranov's constructions has a counterpart for complete collineations. In fact, one piece of the puzzle was already in place: Vainsencher had already shown in 1984 that the complete collineations could be obtained by a blow-up construction very similar to one given by Kapranov for $\overline{M}_{0,n}$. But all the other pieces also fit neatly. For example, $\overline{M}_{0,n}$ is constructed by Kapranov as the Chow quotient of a Grassmannian by a $(\mathbb{C}^{\times})^n$ -action; likewise, the complete collineations are constructed here as the Chow quotient of a Grassmannian by a \mathbb{C}^{\times} -action.

(0.1) Outline of the paper. Section 1 of the paper reviews the quotient constructions mentioned above, and Kapranov's work on $\overline{M}_{0,n}$. Section 2 then reviews Vainsencher's work and states our main theorem, which closely parallels Kapranov's.

Sections 3 and 4 are an apparent digression on the varieties obtained as Mumford quotients of the Grassmannian by the aforementioned \mathbb{C}^{\times} -action. It turns out that they form a sequence of smooth varieties obtained from a projective space by blowing up and down the secant varieties of the Segre embedding, much like the author's previous work [31, 32, 33]. This identifies the inverse limit of Mumford quotients with Vainsencher's construction, since

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his blow-up loci are the proper transforms of the secant varieties.

Sections 5 through 8 proceed with the proof of the main theorem. One key step, described in §6, is a version of the Gel'fand-MacPherson correspondence relating quotients by a reductive group to quotients by a torus. Another is the construction of a morphism from the Chow quotient to any Mumford quotient; this is due to Kapranov but re-proved in §7.

Section 9 is a brief remark explaining how the complete collineations can also be interpreted as a space of stable maps to a Grassmannian, and how this can be used to evaluate certain Gromov-Witten invariants.

The final three sections are appendices. Sections 10 and 11 deal with the special case when the two vector spaces in the complete collineations are dual to each other. In this case one can restrict to the symmetric and anti-symmetric loci, to obtain smaller spaces: the classical space of complete quadrics, and the space of complete skew forms. Everything stated for complete collineations has a symmetric and anti-symmetric analogue in an appealing way. For example, the Grassmannian becomes a Lagrangian or orthogonal Grassmannian, and the Segre embedding becomes a quadratic Veronese or Plücker embedding.

Section 12, although it discusses the original motivation for the paper, has been banished to the end because it breaks the flow. It explains how the Chow quotient at the heart of the paper may be understood in terms of a familiar object in symplectic geometry, namely the space of broken Morse flows of the moment map for a circle action. Nothing from this section is needed elsewhere in the paper, but it sheds some light on the geometric intuition behind the Chow quotient.

(0.2) Conventions. The base field is the field of complex numbers, or any algebraically closed field of characteristic 0. The morphism from the Hilbert scheme to the Chow variety requires characteristic 0, but many of the results not involving the Hilbert scheme should hold in arbitrary characteristic.

For a vector space U, $\operatorname{Gr}_i U$ denotes the Grassmannian of i-dimensional subspaces of U. The tautological subbundle and quotient bundle over $\operatorname{Gr}_i U$ are denoted S_U and Q_U , respectively. Also, $\mathbb{P}U$ denotes the space of lines in U, so that $\mathbb{P} = \operatorname{Gr}_1$.

The notation $X/\!\!/G$, $X/\!\!/G$, $X/\!\!/G$, for Hilbert, Chow and Mumford quotients respectively, follows Kapranov [15], but it conflicts with many other papers where $X/\!\!/G$ denotes a Mumford quotient.

The word *unstable* is used to mean "not semistable".

The term *complete collineation* is reserved by Laksov [22], but not Vainsencher [35], for the case where the two vector spaces are isomorphic. Despite the loss of precision, we prefer *collineation* for its euphony.

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After this paper was written, I learned from M. M. Kapranov of the previous work of Yu. A. Neretin. This includes a result similar to the identification of (1) and (5b) in the Main Theorem. However, Neretin's work is not in algebraic geometry, but rather in the category of metric spaces.

1 Review of algebraic quotient constructions

(1.1) Hilbert quotients. Let G be an algebraic group acting on a projective variety X. There is an open set $U \subset X$ such that the orbit closures of points in U form a flat family of subschemes of X. Indeed, let $\Delta \subset X \times X$ be the diagonal; the closure $Y = \overline{G \cdot \Delta}$ is a projective variety, with a morphism $Y \to X$ given by projection on the first factor. For dimension reasons, the fiber of this morphism over a generic $x \in X$ is simply the orbit closure $\overline{G \cdot x}$, an integral subscheme of X. Since X is integral, there is a nonempty open set $U \subset X$ such that $Y|_U$ is flat [26, §8].

Hence there is a morphism $U \to \mathcal{H}_X$, where \mathcal{H}_X is the Hilbert scheme parametrizing subschemes of X. Define the *Hilbert quotient* $X/\!\!/\!\!/ G$ as the closure of the image of U in \mathcal{H}_X . This is a projective variety, and is independent of U. It parametrizes a family of orbit closures, together with their flat limits as schemes.

(1.2) Chow quotients. The family of schemes over the open set U above can also be viewed as a family of algebraic cycles. Since U is reduced, this determines a morphism from U to the Chow variety parametrizing algebraic cycles on X of the appropriate intersection class [19, I 3.10, 3.21]. Define the Chow quotient $X/\!\!/ G$ as the closure of the image of U in the Chow variety. Again, this is a projective variety, and is independent of U. It parametrizes a family of orbit closures, together with their limits as cycles.

In fact, Mumford [27, §5.4] showed that in characteristic 0, there is a morphism from the induced reduced subscheme of every component of the Hilbert scheme to the appropriate Chow variety. Hence there is a canonical morphism from the Hilbert quotient to the Chow quotient.

(1.3) Inverse limits of Mumford quotients. Suppose now that X is normal and G is reductive. Geometric invariant theory associates a Mumford quotient X/G(L) to every ample (or even base-point free) linearization L of the G-action. Here a linearization is a line bundle on X endowed with a linear action of G lifting the action on X. The quotient is defined to be $\operatorname{Proj} \bigoplus_k H^0(L^k)^G$, where the superscript denotes the G-invariant part. It is the categorical quotient of an open subset of X, called the semistable set.

The Mumford quotients are familiar, but depend on the non-canonical choice of a linearization. Less familiar, but more canonical, is their inverse limit, which will be our focus of interest.

The Mumford quotients are given the structure of an inverse system in the following way. First, let the G-Néron-Severi group $\mathrm{NS}^G(X)$ be the group of G-algebraic equivalence classes of linearizations on X, where G-algebraic equivalence is the obvious generalization of ordinary algebraic equivalence to linearizations. Then there is a short exact sequence

$$0 \longrightarrow \chi(G) \otimes \mathbb{Q} \longrightarrow \mathrm{NS}^G(X) \otimes \mathbb{Q} \longrightarrow \mathrm{NS}(X) \otimes \mathbb{Q} \longrightarrow 0,$$

where $\chi(G) = \operatorname{Hom}(G, \mathbb{C}^{\times})$. Moreover, up to isomorphism the quotient X/G(L) depends only on the class of L in $\mathbb{P}(\operatorname{NS}^G(X) \otimes \mathbb{Q})$. In a slight abuse of terminology, an element of this rational projective space will be called a *fractional linearization*. Second, define a chamber structure on a domain in a rational projective space to be a partition that is locally a linear projection of the faces of a rational polytope. Then there is a chamber structure on the set of base-point free linearizations in $\mathbb{P}(\operatorname{NS}^G(X) \otimes \mathbb{Q})$, having only finitely many chambers, such that up to isomorphism, the quotient X/G(L) depends only on the chamber containing L. Third, taking closures gives the set of chambers the structure of a directed set. For any arrow in this directed set, there is a natural morphism of the corresponding quotients. This makes the collection of all nonempty Mumford quotients X/G(L) into an inverse system. Because the chambers are finite in number, the inverse limit is a projective scheme of finite type over \mathbb{C} .

The statements in the previous paragraph are proved in the works of Dolgachev-Hu [7] and the author [33]; in the torus case, many of them are due to Brion-Procesi [5]. However, only one of them is substantially difficult to prove, namely the finiteness of the chambers. Everything else in some sense follows from the Hilbert-Mumford numerical criterion [27]. Moreover, in the cases treated herein, the finiteness is also fairly easy. So the inverse limits we need can be defined without invoking the general theory.

The Hilbert quotient, the Chow quotient, and the limit of the Mumford quotients are all canonically associated to a normal variety with a reductive group action. In the case of torus actions on toric varieties, they have been studied by Kapranov-Sturmfels-Zelevinsky [17], where they are generally all different. In particular, the Chow and Hilbert quotients are irreducible by definition, while the inverse limit may be reducible even in a very simple example [33, 1.11]. However, in any case the limit of Mumford quotients has a distinguished component containing the quotients of that open set of points which are semistable whenever the quotient is nonempty.

Kapranov [15, 16] went on to make a detailed study of these quotients in a specific case. We conclude this section by stating his results, which were a principal inspiration for the present work. They give several remarkable constructions of the moduli space of stable curves of genus 0.

For any n, let Γ_n be a set of n points in general position in \mathbb{P}^{n-2} . Note that all such sets are projectively equivalent. Define varieties $X_{n,k}$ recursively as follows: $X_{n,1} = \mathbb{P}^{n-3}$, and $X_{n,k+1}$ is the blow-up of $X_{n,k}$ along the proper transforms of all the k-1-planes spanned by k points in Γ_{n-1} . Kapranov [15] is very careful about the order in which these k-planes are blown up. However, as D. Thurston has pointed out, when two smooth subvarieties are transverse, the order in which their proper transforms are blown up does not matter. All the k-planes in question are pairwise transverse, so blowing them up in any order yields the same variety.

(1.4) Theorem (Kapranov). The following are isomorphic:

- (1) The moduli space $\overline{M}_{0,n}$ of stable n-punctured curves of genus 0;
- (2) The variety $X_{n,n-2}$ defined above;
- (3) The closure (a) in the Hilbert scheme, or (b) in the Chow variety of \mathbb{P}^{n-2} , of the locus of rational normal curves containing Γ_n ;

- (4) The (a) Chow quotient, or (b) inverse limit of Mumford quotients $(\mathbb{P}^1)^n/\mathrm{SL}(2)$;
- (5) The (a) Hilbert quotient, or (b) Chow quotient, or (c) inverse limit of Mumford quotients $\operatorname{Gr}_2 \mathbb{C}^n / (\mathbb{C}^{\times})^n$, where $(\mathbb{C}^{\times})^n$ acts in the natural way.

2 Complete collineations

(2.1) **Definitions.** The spaces of complete collineations were introduced and extensively studied at the end of the nineteenth century. The literature on them is vast and we content ourselves with a very sketchy review. A comprehensive history, both ancient and modern, is recounted by Laksov [21, 22].

Let V, W be finite-dimensional vector spaces, and let $u \leq \min(\dim V, \dim W)$. Over $\operatorname{Gr}_u V$, there is a rational map

$$\mathbb{P}\operatorname{Hom}(S_V,W) \dashrightarrow \underset{i=0}{\overset{u}{\times}} \mathbb{P}\operatorname{Hom}(\Lambda^i S_V,\Lambda^i W)$$

given by $f \mapsto (\Lambda^i f)$. The space of rank u complete collineations is defined to be the closure of the image. By definition, this compactifies the space of isomorphisms between u-dimensional subspaces of V and W, up to scalars. When $u = \dim V = \dim W$, for example, it is a compactification of $\operatorname{PGL}(u)$.

The name given to this space suggests that there should exist an object called a rank u complete collineation of which it is the moduli space. This is indeed true: a rank u complete collineation $V \to W$ is a u-dimensional subspace U of V, together with a finite sequence of nonzero linear maps f_i , where $f_1: U \to W$, $f_{i+1}: \ker f_i \to \operatorname{coker} f_i$, and the last f_i has maximal rank.

However, the notion of families of such objects is rather delicate; it is meticulously treated by Kleiman-Thorup [18] and Laksov [23]. Roughly speaking, where f_i drops rank, its derivative should be f_{i+1} . Rather than struggle with this interpretation, we will make contact with the complete collineations via an explicit construction of the moduli space, due to Vainsencher in 1984 [35].

(2.2) Vainsencher's construction. Given V, W, and u as above, define a sequence of varieties $\mathcal{X}_k^u(V,W)$, or just \mathcal{X}_k for short, by the following recursion. Let \mathcal{X}_1 be the projective bundle $\mathbb{P} \operatorname{Hom}(S_V,W)$ mentioned above, and let \mathcal{X}_k be the blow-up of \mathcal{X}_{k-1} at the proper transform of the locus of linear maps $S_V \to W$ of rank k-1. Vainsencher proved that $\mathcal{X}_u^u(V,W)$ is isomorphic to the space of rank u complete collineations.

If $u = \dim V$, for example, then \mathcal{X}_1 is just the projective space $\mathbb{P}\operatorname{Hom}(V, W)$, and the blow-up loci are the proper transforms of the secant varieties of the Segre embedding $\mathbb{P}V^* \times \mathbb{P}W \hookrightarrow \mathbb{P}\operatorname{Hom}(V, W)$.

The similarity of this recursive construction to Kapranov's is striking. Our main result shows that this analogy goes much further.

(2.3) Main Theorem. The following are isomorphic:

- (1) The moduli space of rank u complete collineations $V \to W$;
- (2) The variety $\mathcal{X}_{u}^{u}(V, W)$ defined above;
- (3) The closure (a) in the Hilbert scheme, or (b) in the Chow variety of $\mathbb{P}V \times \mathbb{P}W$, of the locus of graphs of linear injections $U \to W$, where $U \subset V$ ranges over all u-dimensional subspaces;
- (4) The (a) Chow quotient, or (b) inverse limit of Mumford quotients $\mathbb{P} \operatorname{Hom}(U, V^*) \times \mathbb{P} \operatorname{Hom}(U, W) / \operatorname{SL}(U)$, where U is a fixed vector space of dimension u;
- (5) The (a) Hilbert quotient, or (b) Chow quotient, or (c) inverse limit of Mumford quotients $\operatorname{Gr}_u(V \oplus W)/\mathbb{C}^{\times}$, where \mathbb{C}^{\times} acts with weight 1 on V and -1 on W.

As we have seen, the equivalence of (1) and (2) is Vainsencher's, but the rest appears to be new. The proof of the remaining statements occupies sections 3 through 8. The diagram below shows which equivalences are proved in which sections.

3 Quotients of the Grassmannian by the multiplicative group

As in §2 above, V, W are finite-dimensional vector spaces, and u is a positive integer such that $u \leq \min(\dim V, \dim W)$. Let $T = \mathbb{C}^{\times}$ act on $V \oplus W$ by $\lambda(v, w) = (\lambda v, \lambda^{-1}w)$; this induces a T-action on $Y = \operatorname{Gr}_u(V \oplus W)$. This section concerns the Mumford quotients of Y by this action. In fact, all the constructions are sufficiently natural that V and W could, more generally, be vector bundles over a common base.

The T-action is canonically linearized on the ample line bundle $\det Q_{V \oplus W}$. However, the canonical linearization is not the only one, because the group of characters $\chi(T)$ is not trivial, but isomorphic to \mathbb{Z} . For any $t \in \chi(T) \otimes \mathbb{Q} \cong \mathbb{Q}$, the canonical linearization L can be tensored by the fractional character t to produce a new fractional linearization L(t). Indeed, by the short exact sequence of (1.3), any ample linearization is a positive tensor power of some L(t).

For $\sigma \in \mathbb{Q}$, say $U \in Y$ is σ -semistable if it is semistable for the linearization $L(u-2\sigma)$, and let Y_{σ} be the corresponding Mumford quotient. We aim to compare the quotients for different σ , using the results of [14, 33]. Actually, in the torus case those results are only slight refinements of the work of Brion-Procesi [5].

A straightforward application of the Hilbert-Mumford numerical criterion [27] yields the following result.

- (3.1) Proposition. A point $U \in Y$ is
 - (a) σ -semistable if and only if dim $U \cap V \leq u \sigma$ and dim $U \cap W \leq \sigma$;
 - (b) σ -stable if and only if both inequalities are strict;
 - (c) identified in the quotient with $(U \cap V) \oplus (U + V)/V$ if equality holds in the first case, and with $(U + W)/W \oplus (U \cap W)$ if it holds in the second.

(3.2) Corollary.

- (a) For $\sigma \notin [0, u]$, $Y_{\sigma} = \emptyset$;
- (b) For $\sigma \in (0,1)$, Y_{σ} is the projective bundle $\mathbb{P} \operatorname{Hom}(S_V,W)$ over $\operatorname{Gr}_u V$;
- (c) For $\sigma \in (u-1,u)$, Y_{σ} is the projective bundle $\mathbb{P} \operatorname{Hom}(S_W,V)$ over $\operatorname{Gr}_u W$.

The semistability condition, and hence the quotient Y_{σ} , is locally constant for $\sigma \notin \mathbb{Z}$. At an integer value, on the other hand, the semistability condition changes. The chamber decomposition of (1.3) therefore consists simply of the integer points in [0, u] and the intervals between them. The quotients for $\sigma \in [1, u - 1]$ can then be described using Theorem 4.8 of [33], which explains how the quotient changes when a wall is crossed. Its hypotheses are readily verified, and it directly implies the theorem below.

Fix an integer $k \in [1, u-1]$. Write Y_k^- for $Y_{k-\frac{1}{2}}$ and Y_k^+ for $Y_{k+\frac{1}{2}}$. Also let Z_k^0 be the locus in Y_k consisting of points unstable for $\sigma \neq k$, let Z_k^- be the locus in Y_k^- consisting of points unstable for $\sigma > k$, and let Z_k^+ be the locus in Y_k^+ consisting of points unstable for $\sigma < k$.

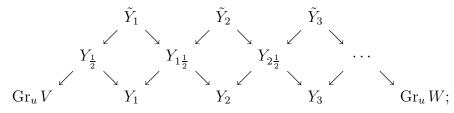
- (3.3) Theorem. With the above notation,
 - (a) Z_k^{\pm} are naturally projective bundles over Z_k^0 : indeed there are natural isomorphisms

$$Z_k^0 = \operatorname{Gr}_{u-k} V \times \operatorname{Gr}_k W,$$

 $Z_k^- = \mathbb{P} \operatorname{Hom}(S_W, Q_V),$
 $Z_k^+ = \mathbb{P} \operatorname{Hom}(S_V, Q_W);$

- (b) the birational morphisms of quotients $Y_k^- \to Y_k$ and $Y_k^+ \to Y_k$ induced by the inclusions $Y^{ss}(k\pm \frac{1}{2}) \subset Y^{ss}(k)$ restrict to the projections $Z_k^\pm \to Z_k^0$, and are isomorphisms elsewhere;
- (c) the fibered product $Y_k^- \times_{Y_k} Y_k^+$ is naturally isomorphic to the blow-up of Y_k^- along Z_k^- , and to the blow-up of Y_k^+ along Z_k^+ .

There is therefore a diagram of morphisms, all birational except in the bottom corners:



here \tilde{Y}_i denotes the fibered product, and the morphisms in the top row are the blow-ups of Z_k^{\pm} .

4 Secant variety interpretation

The results of the previous section have an appealing geometric interpretation. Denote by $\operatorname{Sec}_k(S_V,W)$ the locus in $Y_{\frac{1}{2}}=\mathbb{P}\operatorname{Hom}(S_V,W)$ consisting of linear maps of rank $\leq k$. This is precisely the variety of secant (k-1)-planes to the relative Segre embedding of $\mathbb{P}S_V^*\times\mathbb{P}W$ in $\mathbb{P}(S_V^*\otimes\mathbb{P}W)$.

(4.1) **Proposition.** The blow-up locus Z_k^- is the proper transform of $Sec_k(S_V, W)$.

Proof. By definition Z_k^- consists of (quotients of) points unstable for all $\sigma > k$. By Proposition 3.1 this is the quotient of the locus of subspaces $U \in Y$ satisfying $\dim(U \cap V) \ge u - k$. On the other hand, $Y^{ss}(\frac{1}{2})$ is simply the locus of graphs of linear maps from a u-dimensional subspace of V to W. These two loci meet in the locus of such graphs of rank $\le k$. Its quotient $\operatorname{Sec}_k(S_W, V)$ is therefore the proper transform of Z_k^- in $Y_{\frac{1}{n}}$.

A convenient example to keep in mind is when $\dim V = u$, so that $\operatorname{Gr}_u V$ is a point. Then $Y_{\frac{1}{2}}$ is simply the projective space $\mathbb{P}\operatorname{Hom}(V,W)$, and the locus of maps of rank $\leq i$ is the variety of secant (i-1)-planes to the image of the Segre embedding $\mathbb{P}V^* \times \mathbb{P}W \hookrightarrow \mathbb{P}\operatorname{Hom}(V,W)$. Theorem 3.3 therefore shows that the proper transforms of these secant varieties can be blown up and down in turn, to produce a sequence of smooth varieties. When $\dim W = \dim V$ as well, then $Y_{u-\frac{1}{2}}$ is also a projective space, and the rational map $Y_{\frac{1}{2}} \dashrightarrow Y_{u-\frac{1}{2}}$ is the Cremona transformation given by inversion of the linear map; the diagram factors it into a sequence of blow-ups and blow-downs.

There are several other cases where the secant varieties of some variety M in projective space may be blown up and down in this way. For example, according to the author's previous work [31], it is possible when M is a smooth curve of genus g embedded by a complete linear system of degree > 2g - 2. It is also possible when M consists of < n + 1 [15, 32] or n + 2 [33, §6] general points in \mathbb{P}^n . And we will see in §§10 and 11 that it is possible for the quadratic Veronese embedding of \mathbb{P}^n and for the Plücker embedding of $\mathrm{Gr}_2 \mathbb{C}^n$, just by taking appropriate cross-sections of the Segre embedding.

It is quite an interesting and challenging problem to understand how generally this phenomenon occurs. On the one hand, the list assembled above looks promising, but on the other hand, simple examples (such as that of 6 general points in \mathbb{P}^3) show that the varieties involved may not be projective. In that case, the construction of blow-downs becomes extremely problematic. We hope to return to this fascinating subject in the future.

5 The inverse limit of Mumford quotients of the Grassmannian

The Main Theorem is concerned not with any single Mumford quotient, but with their inverse limit. In the case of the action of $T = \mathbb{C}^{\times}$ on $Y = \operatorname{Gr}_u(V \oplus W)$ from §3, this can be constructed as follows.

For any positive integer k < u, define a sequence of varieties $\mathcal{Y}_k^u(V, W)$, or just $\mathcal{Y} - k$ for short, and morphisms $\phi_k : \mathcal{Y}_k \to Y_k$ by the following recursion. Let \mathcal{Y}_1 be $Y_{\frac{1}{2}}$, and given \mathcal{Y}_k , let \mathcal{Y}_{k+1} be the fibered product

$$\mathcal{Y}_k \times_{Y_k} Y_{k+\frac{1}{2}}$$
.

Then $\mathcal{Y}_u^u(V, W)$ is precisely the inverse limit of all the Mumford quotients, and will shortly be shown isomorphic to the space of complete collineations.

The intricate geometry of these spaces arises from a recursive structure: each one contains similar spaces of lower dimension.

(5.1) Proposition. Over $Z_k^0 = \operatorname{Gr}_{u-k} V \times \operatorname{Gr}_k W$, the restriction of ϕ_k is the projection from $\mathcal{Y}_k^k(Q_V, S_W)$, that is, the locally trivial family whose fiber over (V', W') is $\mathcal{Y}_k^k(V/V', W')$.

Proof. First let us determine $\phi_k^{-1}(V',W')$ for any single element $(V',W')\in Z_k^0$. Let

$$Y' = \{ U \in Y \mid V' \subset U \subset V \oplus W' \}.$$

This is a subvariety of Y naturally isomorphic to $\operatorname{Gr}_k(V/V'\oplus W')$. Proposition 3.1 implies that the quotient $Y'_{\sigma}\subset Y_{\sigma}$ is nonempty if and only if $\sigma\in[0,k]$, equals the single point (V',W') if $\sigma=k$, and is preserved by taking the image in the natural maps $Y_k^{\pm}\to Y_k$, as well as the *inverse* image in one direction $Y_k^-\to Y_k$ only. It immediately follows that $\phi_k^{-1}(V',W')=\mathcal{Y}_k^k(V/V',W')$.

To show that the collection of these spaces forms the locally trivial family in the statement is straightforward: just consider, in place of Y', the locally trivial family of Grassmannians $\operatorname{Gr}_k(Q_V \oplus S_W)$ parametrized by Z_k^0 and the natural map from this family to Y.

In the Morse-theoretic terms outlined in §12, Y' is the closure of the upward Morse flow from (V', W').

(5.2) Proof that (5c) = (2). We now show that the inverse limit $\mathcal{Y}_u^u(V, W)$ constructed above is isomorphic to Vainsencher's blow-up $\mathcal{X}_u^u(V, W)$.

By definition $\mathcal{X}_1 \cong \mathcal{Y}_1$. By induction on k, it suffices to assume $\mathcal{X}_k \cong \mathcal{Y}_k$ and prove $\mathcal{X}_{k+1} \cong \mathcal{Y}_{k+1}$. Now \mathcal{X}_{k+1} is the blow-up of \mathcal{X}_k along the proper transform of the locus of rank k linear maps, which by Proposition 4.1 is the proper transform of Z_k^- in \mathcal{X}_k . On the other hand, by Theorem 3.3 the fibered product $Y_{k-\frac{1}{2}} \times_{Y_k} Y_{k+\frac{1}{2}}$ is the blow-up of $Y_{k-\frac{1}{2}}$ at Z_k^- . Taking the fibered product with \mathcal{Y}_{k-1} over Y_{k-1} shows that \mathcal{Y}_{k+1} is the blow-up of \mathcal{Y}_k at $\mathcal{Y}_{k-1} \times_{Y_{k-1}} Z_k^-$, which is the total transform of Z_k^- in \mathcal{Y}_k . It therefore suffices to show that the total transform equals the proper transform, or equivalently, that the total transform is an integral scheme.

By Theorem 3.3, Z_k^- is the inverse image of Z_k^0 in the natural map $Y_{k-\frac{1}{2}} \to Y_k$. Its total transform is therefore nothing but $\phi_k^{-1}(Z_k^0)$. But this was shown in Proposition 5.1 to be a locally trivial fibration over $\operatorname{Gr}_{u-k} V \times \operatorname{Gr}_k W$ with fiber $\mathcal{Y}_k^k(V/V', W')$. By induction on u, this may be assumed to be integral, and that completes the proof.

6 The Gel'fand-MacPherson correspondence

To relate the quotients of the Grassmannian to those of $\mathbb{P} \operatorname{Hom}(U, V) \times \mathbb{P} \operatorname{Hom}(U, W)$, we use a version of the so-called *Gel'fand-MacPherson correspondence* [10]. The argument is also reminiscent of the author's trick [33, 3.1] for converting a quotient by a reductive group to one by a torus. It is carried out here first for Mumford quotients, then for Chow quotients.

(6.1) Proof that (5c) = (4b). Let $T = \mathbb{C}^{\times}$ act on $Y = \operatorname{Gr}_{u}(V \oplus W)$ as before. We wish to show that the inverse limit of Mumford quotients Y/T is isomorphic to that of $\mathbb{P}\operatorname{Hom}(U,V) \times \mathbb{P}\operatorname{Hom}(U,W)/\operatorname{SL}(U)$. It suffices to show that the inverse systems of Mumford quotients are isomorphic. We will see that both are isomorphic to the inverse system of Mumford quotients $\mathbb{P}\operatorname{Hom}(U,V \oplus W)/(T \times \operatorname{SL}(U))$.

There is an unique ample linearization of the $\mathrm{SL}(U)$ -action on $\mathbb{P} \mathrm{Hom}(U,V\oplus W)$, and the Mumford quotient is the Grassmannian Y. It is an easy exercise to check that the descent on linearizations induces an isomorphism

$$NS_{\mathbb{O}}^{T}(Y) \cong NS_{\mathbb{O}}^{T \times SL(U)} (\mathbb{P} \operatorname{Hom}(U, V \oplus W)).$$

It follows from the definition of Mumford quotients that

$$\mathbb{P}\operatorname{Hom}(U, V \oplus W)/(T \times \operatorname{SL}(U)) \cong Y/T$$

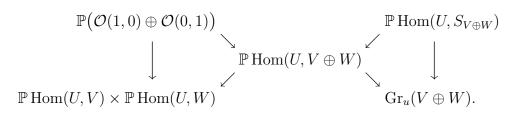
naturally, if the linearization on Y is taken to be the descent of that on the projective space. Indeed, the isomorphism is induced by a morphism of the corresponding semistable sets. The morphisms in the inverse systems are induced by inclusions of semistable sets, so they commute with these isomorphisms. Hence the inverse systems of Y/T and of $\mathbb{P}\operatorname{Hom}(U,V\oplus W)/(T\times\operatorname{SL}(U))$ are isomorphic.

Similar arguments apply to the T-action on $\mathbb{P}\operatorname{Hom}(U,V\oplus W)$. The linearization is no longer unique, since it can be twisted by a character of T. But every nonempty quotient is naturally isomorphic to $\mathbb{P}\operatorname{Hom}(U,V)\times\mathbb{P}\operatorname{Hom}(U,W)$. (Actually, this is not strictly true, as the special cases $\sigma=0$ and $\sigma=u$ have just $\mathbb{P}\operatorname{Hom}(U,V)$ and $\mathbb{P}\operatorname{Hom}(U,W)$ as their respective quotients. However, it does no harm to pretend that the quotient remains $\mathbb{P}\operatorname{Hom}(U,V)\times\mathbb{P}\operatorname{Hom}(U,W)$, since further dividing by the $\operatorname{SL}(U)$ -action gives $\operatorname{Gr}_u V$ and $\operatorname{Gr}_u W$ in either case.) The descent again induces an isomorphism on the space of fractional linearizations. Hence the inverse systems of $\mathbb{P}\operatorname{Hom}(U,V\oplus W)/(T\times\operatorname{SL}(U))$ and of $\mathbb{P}\operatorname{Hom}(U,V)\times\mathbb{P}\operatorname{Hom}(U,W)/\operatorname{SL}(U)$ are isomorphic. \maltese

(6.2) Lemma. Let A be a cycle on $M \times N$ such that for any $x \in N$, the naive intersection $A \cap (M \times \{x\})$ has the expected dimension. Then A determines a morphism from N to the appropriate Chow variety of M.

Proof. This is proved by Samuel [30, II 5.12, 6.8], or, in more modern language, by Kollár [19, I 3.10, 3.21]. \maltese

(6.3) Proof that (5b) = (4a). Consider the diagram



Here $\mathbb{P}(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ refers to the \mathbb{P}^1 -bundle over $\mathbb{P}\operatorname{Hom}(U,V) \times \mathbb{P}\operatorname{Hom}(U,W)$, and the diagonal arrows in the second row are only rational maps, geometric quotients of open subsets by T and $\operatorname{SL}(U)$, respectively.

Certainly the universal cycle on $(Y/\!\!/T) \times Y$ satisfies the lemma. It may be pulled back by the flat morphism in the right-hand column. The pulled-back cycle on $(Y/\!\!/T) \times \mathbb{P} \operatorname{Hom}(U, S_{V \oplus W})$ again satisfies the lemma, and over a general point in $Y/\!\!/T$, its fiber is an $\operatorname{SL}(U) \times T$ -orbit closure. This cycle can be pushed forward to $(Y/\!\!/T) \times \mathbb{P} \operatorname{Hom}(U, V \oplus W)$. The pushed-forward cycle satisfies the lemma, since a birational morphism does not increase the codimension of any subvariety. But it is also T-invariant, so it descends to a cycle on the geometric quotient $Y/\!\!/T \times \mathbb{P} \operatorname{Hom}(U, V) \times \mathbb{P} \operatorname{Hom}(U, W)$. This again satisfies the lemma, hence determines a morphism from $Y/\!\!/T$ to a Chow variety of $\mathbb{P} \operatorname{Hom}(U, V) \times \mathbb{P} \operatorname{Hom}(U, W)$. Over a general point in $Y/\!\!/T$, its fiber is an $\operatorname{SL}(U)$ -orbit closure. Its image is therefore the Chow quotient.

A similar argument using the \mathbb{P}^1 -bundle in the diagram produces a morphism in the opposite direction, which is clearly inverse to the first one generically. Since both spaces are integral, it must be the inverse everywhere.

7 The morphism from Chow to Mumford quotients

The following theorem is essentially that proved by Kapranov [15, 0.4.3], but with a few additions and a different proof.

- (7.1) **Theorem.** Let X be a projective variety, G a reductive group acting on X. For any linearization L with $X^{ss}(L) \neq \emptyset$, and any cycle Y appearing in the Chow quotient $X/\!\!/G$:
 - (a) $Y \cap X^{ss}(L) \neq \emptyset$ and is contained in a unique L-semistable equivalence class of orbits;
 - (b) assigning this class to Y defines a morphism $X/\!\!/G \to X/\!\!/G(L)$, which is birational if $X^s(L) \neq \emptyset$:
 - (c) if L^+ and L^0 are two linearizations such that $X^{ss}(+) \subset X^{ss}(0)$, then the diagram below commutes.

$$\begin{array}{ccc} X/\!\!/G & & & \\ \swarrow & & \searrow & \\ X/G (+) & \longrightarrow & X/G (0) \end{array}$$

Proof. Denote \mathcal{C} the Chow variety containing the orbit closure of a generic point in X. The universal cycle \mathcal{U} on $C \times X$ restricts via Chow pull-back to a universal cycle over $(X/\!\!/ G) \times X$. It is proper, G-invariant, irreducible, has multiplicity 1, and surjects onto both factors. It therefore descends to a cycle D on $(X/\!\!/ G) \times (X^{ss}/G)$. Denote E the open set in X where the orbit closure has the generic homology class. There is a natural morphism $E \to X/\!\!/ G$, but also $E \cap X^{ss} \neq \emptyset$ since both are nonempty and open.

For any $x \in X/\!\!/ G$ in the image of $E \cap X^{ss}$, $D \cap \left(\{x\} \times X^{ss}/G\right)$ clearly consists of a single point. Moreover, for any $x \in X/\!\!/ G$ whatsoever, $D \cap \left(\{x\} \times X^{ss}/G\right)$ is finite: it consists exactly of the semistable equivalence classes of irreducible components of $\mathcal{U} \cap \left(\{x\} \times X^{ss}\right)$. Lemma 6.2 then implies that D is the Chow pull-back of the universal cycle by a morphism from $X/\!\!/ G$ to the Chow variety of 0-dimensional, length 1 cycles in X^{ss}/G , which is of course X^{ss}/G itself. So $D \cap \left(\{x\} \times X^{ss}/G\right)$ consists of exactly one point for any $x \in X/\!\!/ G$ at all. There is hence a unique semistable equivalence class in each cycle of the Chow quotient. Furthermore, the morphism $X/\!\!/ G \to X^{ss}/G$ is clearly an isomorphism over the open set $E \cap X^s/G$, so it is birational if X^s is nonempty. This completes the proof of the first two statements.

The third statement is easy after noting that, since the inclusion $(X/\!\!/ G) \times X^{ss}(+) \subset (X/\!\!/ G) \times X^{ss}(0)$ induces an inclusion of universal cycles, the quotient cycle $D^+ \subset (X/\!\!/ G) \times X/G(+)$ is the inverse image of $D^0 \subset (X/\!\!/ G) \times X/G(0)$.

(7.2) Corollary. For any reductive group action on a projective variety, there exists a morphism from the Chow quotient to the inverse limit of Mumford quotients.

★

At this point the reader may be thinking it likely that the Chow quotient is always isomorphic to the inverse limit of Mumford quotients. In fact, this is far from being true. A counterexample is furnished by the action of SL(2) on $Sym^n \mathbb{P}^1 = \mathbb{P}^n$. The Chow quotient parametrizes those cycles appearing in the quotient of the universal cycle over $\overline{M}_{0,n} \times (\mathbb{P}^1)^n$ by the natural action of the symmetric group Σ_n . It is not hard to see that the Chow quotient differs from both $\overline{M}_{0,n}/\Sigma_n$ and $\mathbb{P}^n/SL(2)$, and indeed is not normal at least when $n \geq 8$. This may be the example alluded to in Remark 0.4.10 of Kapranov [15].

However, let us return to the case at hand: $Y = \operatorname{Gr}_u(V \oplus W)$, $T = \mathbb{C}^{\times}$. It is now easy to describe those cycles belonging to the Chow quotient $Y/\!\!/T$. The proof is straightforward using Proposition 3.1 and the fact, stated in Theorem 7.1, that any cycle in the Chow quotient intersects $X^{ss}(\sigma)$ in exactly one σ -semistable equivalence class, whether σ is integral or not.

(7.3) Proposition. Any cycle appearing in the Chow quotient $Y/\!\!/T$ is a connected, nodal curve of arithmetic genus 0 and degree u, the union of orbit closures of U_1, \ldots, U_i where

$$U_j \neq (U_j \cap V) \oplus (U_j \cap W),$$

$$U_1 \cap W = 0,$$

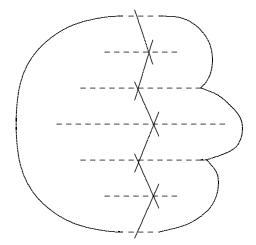
$$U_i \cap V = 0,$$

$$U_{j} \cap V = \left(U_{j+1} + W\right)/W,$$

and $U_{j+1} \cap W = \left(U_{j} + V\right)/V$

for all positive $j \leq i - 1$.

(7.4) Remarks. (1) Of the five equations above, the first is a nondegeneracy condition, the next two say that the curve stretches all the way from one extreme fixed component to the other, and the last two say that the sink of each T-orbit is the source of the next. Three curves satisfying these conditions are illustrated in the diagram at right. The dashed lines represent the components of the fixed-point set. The whole picture is reminiscent of the "broken flows" in Morse theory—an analogy which is expanded upon in §12.



- (2) Each point in the Chow quotient determines flags in V and W by intersecting those spaces with
- the U_i . The Halphen locus (cf. Laksov [22]) is the locus where these are maximal. Equivalently, there are u distinct subspaces U_1, \ldots, U_u giving rise to a cycle which is the union of u lines. This is illustrated by the middle curve in the diagram.
- (3) Finally, the alert reader will notice a direct connection with complete collineations: namely, U_j are the graphs of the linear maps f_j . Thus the Halphen locus is the locus where every f_j has rank 1.
- (7.5) Proof that (5b) = (5c). Let $\mathbb{P}^N = \mathbb{P}\Lambda^u(V \oplus W)$. The Plücker embedding takes Y into \mathbb{P}^N T-equivariantly, and the linearizations $L(\sigma)$ on Y extend naturally over \mathbb{P}^N . Since the extremal weights of the T-action on \mathbb{P}^N are $\pm u$, it follows that the Mumford quotients \mathbb{P}^N_{σ} are nonempty exactly when $\sigma \in [0, u]$: the same range as for Y itself. Hence the inverse limit of the nonempty X_{σ} embeds in that of the \mathbb{P}^N_{σ} .

On the other hand, it is easy to check that a generic T-orbit closure in \mathbb{P}^N has degree u, simply because its source and sink are in $\mathbb{P}\Lambda^u V$ and $\mathbb{P}\Lambda^u W$ respectively. Hence there is a natural embedding of Chow quotients $Y/\!\!/T \subset \mathbb{P}^N/\!\!/T$.

Now the work of Kapranov, Sturmfels, and Zelevinsky [17] on toric varieties shows that the Chow quotient $\mathbb{P}^N/\!\!/T$ is an irreducible component of the inverse limit of Mumford quotients, with the embedding given by Corollary 7.2. Hence the Chow quotient of Y embeds in the inverse limit of its Mumford quotients. But both are varieties of the same dimension, and both are irreducible: the former by definition, the latter by (5.2).

(7.6) Proof that (5b) = (5a). By Proposition 7.3 the Chow quotient parametrizes a family of cycles which are all connected curves of the same arithmetic genus and degree. Regarded as reduced schemes, they therefore all have the same Hilbert polynomial. On the other hand, the Hilbert quotient parametrizes a flat family of schemes, and maps to the Chow quotient by the forgetful morphism taking schemes to cycles. The generic fiber of this family is reduced by definition.

But it follows that every fiber is reduced. Otherwise, it would have a greater Hilbert polynomial than its reduced subscheme. Then the Hilbert polynomial would not be constant in the fibers of the flat family, which is a contradiction [13, III 9.9].

The morphism from the Hilbert quotient to the Chow quotient is therefore injective as well as surjective. Since the Chow quotient is smooth by (5.2) and (7.5), Zariski's main theorem implies that this is an isomorphism.

8 The locus of graphs

This section completes the proof of the Main Theorem by relating the Chow quotient $Y/\!\!/T$ to the locus of graphs of linear maps in $\mathbb{P}V \times \mathbb{P}W$.

Let \mathcal{C} be the Chow variety parametrizing cycles deformation equivalent to the graph in $\mathbb{P}V \times \mathbb{P}W$ of an injective linear map from a u-dimensional subspace of V to W. Then let \mathcal{C}' be the closure in \mathcal{C} of the locus of such graphs. Similarly, let \mathcal{H} be the appropriate Hilbert scheme and \mathcal{H}' the closure of the locus of graphs. The object of this section is to identify both \mathcal{C}' and \mathcal{H}' with the space of complete collineations.

(8.1) Proof that (5b) = (3b). Consider the diagram

$$\begin{array}{ccc} \mathbb{P}S_{V \oplus W} & \longrightarrow & \operatorname{Gr}_u(V \oplus W) \\ \downarrow & & & \\ \mathbb{P}(V \oplus W) & \dashrightarrow & \mathbb{P}V \times \mathbb{P}W. \end{array}$$

The arrow in the bottom row is a rational map, the geometric quotient of an open set by the T-action.

As in (6.3), the universal cycle on $(Y/\!\!/T) \times Y$ can be pulled back, pushed forward and divided by the T-action to produce a cycle on $(Y/\!\!/T) \times \mathbb{P}V \times \mathbb{P}W$. It is easy to see that this cycle satisfies Lemma 6.2 and hence defines a morphism from $Y/\!\!/T$ to \mathcal{C} . The fiber over a point of $Y/\!\!/T$ defined by subspaces U_i as in Proposition 7.3 is taken to the cycle

$$\{[v], [w] \in \mathbb{P}V \times \mathbb{P}W \mid \lambda v \oplus \mu w \in U_i \text{ for some } i \text{ and some } \lambda, \mu \in \mathbb{C}^{\times}\}.$$

In particular, a general orbit closure in Y is taken to the graph of a general linear map $V \to W$. Furthermore, modulo the action of T, the U_i can be recovered from the above set, so the morphism is injective. It therefore remains to show that the derivative is injective.

As seen in Proposition 7.3, every cycle appearing in the Chow quotient Y/T is a nodal curve C. Its normal sheaf N is therefore locally free, and the Zariski tangent space to the Chow variety is $H^0(C, N)$. Hence every nonzero deformation in the Chow quotient is represented by a T-invariant normal vector field, nonzero at the generic point of some component of C.

Such a normal vector field pulls back to the inverse image cycle on $\mathbb{P}S_{V\oplus W}$, pushes forward to $\mathbb{P}V \times \mathbb{P}W$ —at least on the smooth points downstairs—and descends to the quotient by T. On an open subset of the smooth points, it is not killed by the push-forward, because the tangent vectors to Y killed by projection from a point $(U, p \in U) \in \mathbb{P}S_{V\oplus W}$ are only those elements of $T_UY = \text{Hom}(U, (V \oplus W)/U)$ annihilating p.

Thus we obtain a nonzero normal vector field on the smooth points of the cycle Z in $\mathbb{P}V \times \mathbb{P}W$ corresponding to C. This determines an injective map

$$T_C(Y/\!\!/T) \to H^0(Z_{sm}, N_{Z_{sm}/\mathbb{P}V \times \mathbb{P}W}),$$

where Z_{sm} denotes the smooth part of Z.

On the other hand, there is a natural map $T_Z\mathcal{C} \to H^0(Z_{sm}, N_{Z_{sm}/\mathbb{P}V \times \mathbb{P}W})$ which is compatible with the map described above. The derivative $T_C(Y/\!\!/T) \to T_Z\mathcal{C}$ is therefore a factor of an injective map, so it is injective.

(8.2) Proof that (3b) = (3a). The general element of \mathcal{C}' is a single orbit closure, so the natural morphism $\mathcal{H}' \to \mathcal{C}'$ is birational and projective. Now (8.1) has already identified \mathcal{C}' with the complete collineations. An element in the Halphen locus corresponds to

$$\bigcup_{i=1}^{u} \mathbb{P}(U_i + W) / W \times \mathbb{P}(U_i + V) / V = \bigcup_{i=1}^{u} \mathbb{P}^{u-1} \times \mathbb{P}^{i-1}.$$

Regarding this as a reduced scheme, restrict the line bundle $\mathcal{O}(k,k)$ from $\mathbb{P}V \times \mathbb{P}W$ to it; the space of sections then has dimension

$$\sum_{i=0}^{u-1} {u-2-i+k \choose k-1} {i+k \choose k}.$$

This is proved by induction on i, starting with \mathbb{P}^{u-1} and gluing on each remaining component in turn. The component $\mathbb{P}^{u-i} \times \mathbb{P}^{i-1}$ contributes

$$\binom{u-2-i+k}{k-1}\binom{i+k}{k} = \binom{u-1-i+k}{k}\binom{i+k}{k} - \binom{u-2-i+k}{k}\binom{i+k}{k}$$

dimensions to the space of sections, the first term on the right being the dimension of its own sections, and the second term imposing the condition required for sections to agree on the intersection $\mathbb{P}^{u-i} \times \mathbb{P}^{i-2}$.

The general fiber, on the other hand, is just a projective space \mathbb{P}^{u-1} projecting linearly to both $\mathbb{P}V$ and $\mathbb{P}W$. The restriction of $\mathcal{O}(k,k)$ is simply $\mathcal{O}(2k)$, so the space of sections has dimension

$$\dim H^0(\mathbb{P}^{u-1}, \mathcal{O}(2k)) = \binom{u-1+2k}{2k}.$$

This turns out to be exactly the same as the summation above! One can prove it by multiplying both expressions by the formal variable x^{u-1} and summing over u; after straightforward manipulations using the identity

$$\sum_{r=0}^{\infty} \binom{r}{j} x^{j} = \frac{x^{j}}{(1-x)^{j+1}},$$

both generating functions can be identified as $(1-x)^{-1-2k}$. (This is an application of the "Snake Oil" method peddled by Wilf [37].)

The schemes over the Halphen locus in \mathcal{H}' must therefore be reduced. Otherwise, their Hilbert polynomials would exceed those of their reduced subschemes, which equal that of the general fiber by the above. This would contradict flatness [13, III 9.9].

The fiber in \mathcal{H}' over each point in the Halphen locus therefore contains a single closed point. In particular, it is 0-dimensional.

On the other hand, given any point at all in $Y/\!\!/T$, it is easy to find a 1-parameter subgroup of $SL(V) \times SL(W)$ taking it to a limit in the Halphen locus. Just choose any bases for V and W compatible with the flags of (7.4), and then let \mathbb{C}^{\times} act on the ith basis elements of V and W with weights i and -i, respectively. It is straightforward to check that the limiting cycle intersects every component of the fixed-point set, and so must be n the Halphen locus. (By the way, this is the only appearance in our story of the natural $SL(V) \times SL(W)$ -action on complete collineations.)

Of course, $SL(V) \times SL(W)$ acts compatibly on \mathcal{C}' , \mathcal{H}' , and their universal families, so each fiber of $\mathcal{H}' \to \mathcal{C}'$ is taken by some \mathbb{C}^{\times} -action to a limit over the Halphen locus. But in a projective morphism, the dimension of the fiber is semi-continuous. Hence all fibers are 0-dimensional, so the morphism is finite [13, III Ex. 11.2]. Since it is also birational and \mathcal{C}' is smooth, it is an isomorphism by Zariski's main theorem. \maltese

9 Relation with Gromov-Witten invariants

There is still one more interpretation of the space of complete collineations worth mentioning because of its current interest. It relates the complete collineations to stable maps in the sense of Kontsevich [20].

(9.1) Proposition. Let $\overline{M}_{0,2}(Y,u)$ be the moduli space of 2-pointed stable maps of genus 0 and degree u to $Y = \operatorname{Gr}_u(V \oplus W)$, and let $\operatorname{ev} : \overline{M}_{0,2}(Y,u) \to Y \times Y$ be the evaluation map. Then the rank u complete collineations are naturally isomorphic to $\operatorname{ev}^{-1}(\operatorname{Gr}_u V \times \operatorname{Gr}_u W)$.

Proof. It follows from Proposition 7.3 and (7.6) that the Hilbert quotient parametrizes a family of stable maps of genus 0 and degree u, all of which are T-invariant. On the other hand, it is easy to check that these are the only T-invariant maps in $\operatorname{ev}^{-1}(\operatorname{Gr}_u V \times \operatorname{Gr}_u W)$, which is a smooth projective variety [9]. The fixed-point set of the T-action therefore has only one component, so it must be the whole space. (Alternatively, a dimension count will do.)

For simplicity consider the case where $\dim V = u = \dim W$, so that $\operatorname{Gr}_u V$ and $\operatorname{Gr}_u W$ are points. The proposition then asserts simply that the fiber of ev over the point (V, W), and hence by symmetry over a general point, is isomorphic to the complete collineations.

This result can be used to compute some higher-point Gromov-Witten invariants of Grassmannians which, to the author's knowledge, cannot at present be evaluated by other means. One pulls back cohomology classes from Y to the universal Chow family, then pushes forward to Y/T and integrates. For example, the class $c_2(S_{V\oplus W})$ in $H^4(Y,\mathbb{Q})$ vanishes when it is pulled back and pushed forward to Y/T. This can be shown by computing its value on lines in the Halphen locus, which are known [3] to generate H_2 of the complete collineations.

It follows that for any classes $\alpha_1, \ldots, \alpha_k \in H^*(\operatorname{Gr}_u \mathbb{C}^{2u}, \mathbb{Q})$,

$$\langle p, p, c_2, \alpha_1, \dots, \alpha_k \rangle_u = 0.$$

Here $\langle \cdots \rangle_u$ denotes the degree u Gromov-Witten invariant, p is the Poincaré dual of a point, and c_2 is the second Chern class of the tautological bundle, as above.

One could pursue this line of reasoning to evaluate more Gromov-Witten invariants of Grassmannians, and even of Lagrangian and orthogonal Grassmannians using the results of §§10 and 11. However, there are severe constraints on the degrees and classes involved, so the total information available is somewhat limited.

10 Appendix: complete quadrics

The choice of two vector spaces V and W has governed everything done so far. By specializing to the case $W = V^*$, we discover additional structure. Specifically, we can ask the objects of study to be symmetric or anti-symmetric in V.

In fact, historically the study of the symmetric objects, namely the *complete quadrics*, preceded that of complete collineations; it was initiated by Chasles in the 1860's. A complete quadric is a finite sequence of quadrics Q_i , where Q_1 is a hypersurface in $\mathbb{P}V$, Q_{i+1} is a hypersurface in the singular locus of Q_i , and the last Q_i is smooth. This makes sense, because the singular locus of a quadric is always a projective subspace.

As with complete collineations, the notion of a family is delicate, but it reduces to the corresponding notion for a complete collineation if each Q_{i+1} is regarded as a self-adjoint map $\ker Q_i \to \operatorname{coker} Q_i$, up to a scalar.

The moduli space of complete quadrics can be constructed just as in (2.1), only substituting

$$\mathbb{P}\operatorname{Sym}^2 V^* \dashrightarrow \underset{i=0}{\overset{u}{\times}} \mathbb{P}\operatorname{Sym}^2 \Lambda^i V^*$$

for the rational map given there.

Everything asserted so far in the paper goes through mutatis mutandis for complete quadrics.

(10.1) **Theorem.** The following are isomorphic:

- (1) The moduli space of complete quadrics in $\mathbb{P}V$;
- (2) The variety defined by blowing up $\mathbb{P}\operatorname{Sym}^2V^*$ along the image of the Veronese embedding, and then along the proper transforms of each of its secant varieties in turn;
- (3) The closure (a) in the Hilbert scheme, or (b) in the Chow variety of $\mathbb{P}V \times \mathbb{P}V^*$, of the locus of graphs of invertible self-adjoint linear maps $V \to V^*$;
- (4) The (a) Chow quotient, or (b) distinguished component of the inverse limit of Mumford quotients X / SL(U), where U is a fixed vector space of the same dimension as V, and X is the closure in ℙHom(U, V) × ℙHom(U, V*) of the locus of (f, g) with f invertible and f*g self-adjoint;

(5) The (a) Hilbert quotient, or (b) Chow quotient, or (c) inverse limit of Mumford quotients $\text{LaGr}_u(V \oplus V^*)/\mathbb{C}^{\times}$, where LaGr denotes the Grassmannian of subspaces Lagrangian with respect to the natural symplectic form on $V \oplus V^*$, and \mathbb{C}^{\times} acts with weight 1 on V and -1 on V^* .

Sketch of proof. Each step of the proof consists either of exactly imitating the corresponding step in the proof of the Main Theorem, or of exploiting the embedding of an object named here in its counterpart from the Main Theorem. The only subtleties are as follows.

First, one needs to be sure that the Chow quotient and limit of Mumford quotients for the Lagrangian Grassmannian embed in their counterparts for the ordinary Grassmannian. This is accomplished by checking that the degree of a generic T-orbit closure is the same in each, as in (7.5), and by verifying that the smaller quotient is nonempty whenever the larger one is.

Second, one needs to know that the secant varieties of the Veronese embedding are exactly the intersection with $\mathbb{P}\operatorname{Sym}^2V^*$ of the secant varieties of the Segre embedding. This follows from their determinantal description: see for example Harris [12, Ex. 9.19].

Third, in order to be sure that the whole inverse system of Mumford quotients of the Lagrangian Grassmannian comes from the ordinary Grassmannian, one must check that $Pic(LaGr) = \mathbb{Z}$. But this follows from the Borel-Weil theory, since LaGr is a quotient of a semisimple group by a maximal parabolic subgroup.

Finally, one would also like to check that X has Picard group $\mathbb{Z} \oplus \mathbb{Z}$. This would imply that the inverse limit of its Mumford quotients is precisely the space of complete quadrics. However, X has complicated singularities and understanding the Picard group seems intractable. Nevertheless, to conclude only that the distinguished component of the inverse limit coincides with the complete quadrics is easy. After all, the full inverse system of $X/\mathrm{SL}(U)$ has as a subsystem those quotients by linearizations which are restricted from $\mathbb{P}\mathrm{Hom}(U,V)\times \mathbb{P}\mathrm{Hom}(U,V^*)$. The inverse limit therefore maps naturally to the inverse limit of this subsystem, which as a subspace of the inverse limit of $\mathbb{P}\mathrm{Hom}(U,V)\times \mathbb{P}\mathrm{Hom}(U,V^*)/\mathrm{SL}(U)$ is known to be the complete quadrics. On the other hand, by Theorem 7.1 there is also a morphism to the distinguished component from the Chow quotient, which again is known to be the complete quadrics. The distinguished component is therefore sandwiched between the complete quadrics by two birational morphisms whose composite is the identity. \maltese

In fact, Vainsencher, Laksov and others studied the more general notion of a u-complete quadric, defined essentially as a complete quadric in a u-dimensional subspace. This generalization does not fit together with the Chow quotient formulation as well as the rank u complete collineations did, so it will not be discussed here.

11 Appendix: complete skew forms

Although they are not as well documented in the 19th- or 20th-century literature, antisymmetric objects can be studied just as well as symmetric ones. The basic definition is the following: a *complete skew form* is a finite sequence of nonzero 2-forms ω_i , where ω_1 is a 2-form on V, ω_{i+1} is a 2-form on the null space of ω_i , and the last ω_i is either nondegenerate or has 1-dimensional null space.

The moduli space can be constructed as in (2.1), only substituting

$$\mathbb{P}\Lambda^2 V^* \dashrightarrow \underset{i=0}{\overset{u-2}{\times}} \mathbb{P}\Lambda^2 \Lambda^i V^*$$

for the rational map given there.

With this understood, there is also an anti-symmetric counterpart of the Main Theorem.

(11.1) Theorem. The following are isomorphic:

- (1) The moduli space of complete skew forms on V;
- (2) The variety defined by blowing up $\mathbb{P}\Lambda^2V^*$ along the image of the Plücker embedding of $\operatorname{Gr}_2(V^*)$, and then along the proper transforms of each of its secant varieties in turn;
- (3) The closure (a) in the Hilbert scheme, or (b) in the Chow variety of $\mathbb{P}V \times \mathbb{P}V^*$, of either (i) the locus of graphs of invertible skew-adjoint linear maps $f: V \to V^*$, if u is even, or (ii) the locus of subvarieties

$$\{([v], [w]) \in \mathbb{P}V \times \mathbb{P}f(V) \mid f(v) = \lambda w \text{ for some } \lambda \in \mathbb{C}\}$$

for $f: V \to V^*$ skew-adjoint of rank u-1, if u is odd;

- (4) The (a) Chow quotient, or (b) distinguished component of the inverse limit of Mumford quotients Z / SL(U), where U is a fixed vector space of the same dimension as V, and Z is the closure in $\mathbb{P} \operatorname{Hom}(U, V) \times \mathbb{P} \operatorname{Hom}(U, V^*)$ of the locus of (f, g) with f invertible and f^*g skew-adjoint;
- (5) The (a) Hilbert quotient, or (b) Chow quotient, or (c) inverse limit of Mumford quotients $\operatorname{OGr}_u^+(V \oplus V^*)/\mathbb{C}^{\times}$, where OGr denotes the Grassmannian of subspaces orthogonal with respect to the natural symmetric 2-form on $V \oplus V^*$, OGr^+ denotes the connected component containing V, and \mathbb{C}^{\times} acts with weight 1 on V and -1 on V^* .

Sketch of proof. As before, the proof mainly parallels that of the Main Theorem. However, in addition to all of the subtleties mentioned under Theorem 10.1 above, a few new ones appear.

The most salient point is that the orthogonal Grassmannian is reducible: it has two connected components, but only one is needed. We choose the component OGr^+ containing $V \subset V \oplus V^*$. Any element of OGr^+ has even-dimensional intersection with V, so OGr^+ intersects only *every other* component of the fixed-point set. Blow-ups and blow-downs of the quotient therefore appear only when σ crosses an even integer.

The blow-up loci are smooth, and as in Proposition 4.1, the birational map to the first quotient takes them to the secant varieties of the Plücker embedding. They are therefore exactly the proper transforms of these secant varieties.

Once this is taken into account, everything follows the proof for complete collineations. As with complete quadrics, it is better at some points to follow a parallel but independent

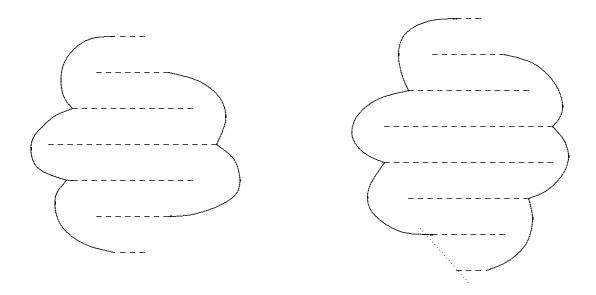
argument, at others to use the embedding of each item in its counterpart from the Main Theorem.

In the case when u is odd, the embeddings differ slightly from the previous cases. For instance, the complete skew forms are not complete collineations in the obvious way, because there are no invertible skew-adjoint linear maps. Nevertheless, they embed in the complete collineations as follows: to each sequence $\omega_1, \ldots, \omega_k$ of skew forms, associate the corresponding skew-adjoint linear maps $f_1: V \to V^*$, $f_2: \ker f_1 \to \operatorname{coker} f_1 = \ker f_1^*$, etc., and then append one final nonzero map $f_{k+1}: \ker f_k \to \operatorname{coker} f_k$, whose choice is unique up to a scalar.

Similarly, all the other constructions also embed in their counterparts from the Main Theorem; it is straightforward to find all these embeddings and show that they are compatible.

 \mathbf{X}

The diagram below portrays typical curves appearing in the Chow quotient of the orthogonal Grassmannian. The case u even is shown on the left; the case u odd, on the right. The dashed lines represent the fixed components of the T-action on the full Grassmannian. Two curves are shown in each case: the first contained in the good component OGr^+ and the second contained in the other discarded component. Notice that they only visit every other fixed component. In the even case, the good curve stretches all the way from one extreme fixed component to the other, so it actually comes from the Chow quotient of the full Grassmannian. In the odd case, even the good curve does not make it all the way, so an extra line (shown in dots) must be added to define an embedding of $\operatorname{OGr}^+/\!/T$ in $\operatorname{Gr}/\!/T$.



An alternate construction of a space of complete skew forms has been given by Bertram [2]. Despite his different approach using Pfaffians, he obtains what appear to be the same spaces as those discussed here.

12 Appendix: symplectic quotients and broken Morse flows

At least heuristically, the Chow quotient of the Grassmannian may be understood in terms of broken Morse flows. Without proving anything, this final section will explain that analogy.

Let X be a smooth compact manifold with a symplectic form ω , and a Hamiltonian action of the Lie group U(1). Here Hamiltonian means that the action preserves ω , and admits a $moment\ map$, namely a primitive function for the infinitesimal action in the following sense. The infinitesimal action of U(1) determines a vector field $\xi \in \Gamma(TX)$. A moment map is a smooth map $\mu: X \to \mathbb{R}$ such that $d\mu = \omega(\xi, \cdot)$ as 1-forms. Assume for simplicity that the action is quasi-free, meaning that it has no nontrivial finite stabilizers. If t is a regular value of μ , then the $symplectic\ quotient$ or $reduced\ phase\ space\ \mu^{-1}(t)\ /\ U(1)$ is a manifold, which turns out to have a natural symplectic structure. (It first arose in classical mechanics, where X was a phase space, the group action reflected some symmetry of the system, and the moment map was the associated conserved quantity.)

Several authors [11, 14] have studied the relationship between the symplectic quotients for different values of t. A key point is that μ is a Morse function in the sense of Bott [4]. Indeed, the critical manifolds are precisely the fixed manifolds of the U(1)-action. The different level sets are therefore related by a sequence of surgeries: submanifolds diffeomorphic to sphere bundles over the critical manifolds are removed, and replaced by sphere bundles of different dimension. The U(1)-action preserves these bundles, and indeed acts in the standard way on the odd-dimensional spheres which are the fibers. Dividing by this action produces bundles with fiber a complex projective space, so the quotients are related by excising one projective bundle and replacing it with another.

Those who prefer a quotient to be a canonical object may be unhappy that the symplectic quotient depends on the choice of t. They will prefer the following alternative. Let F be the space of Morse flow lines from the absolute maximum of μ to its absolute minimum. This is a manifold which can clearly be identified with a dense open subset of every level set, simply by intersecting the level set with the flow. It is well-known in Morse theory [1] that F can be compactified by adding the so-called broken flows. These are unions of flow lines, each beginning at the same critical point where the previous one ends. The picture is exactly like the one shown in §7.

The space \overline{F} of broken flows is not generally a manifold, but only a manifold with corners. The U(1)-action on \overline{F} is free, so the quotient is again a manifold with corners. But each stratum is foliated by tori coming from the U(1) actions on the components of the broken flow: the smaller the stratum, the larger the torus. If these tori are all collapsed, then the resulting space \mathcal{X} is a symplectic manifold. It parametrizes broken flows modulo an equivalence in which each component of the flow rotates separately. The proof that \mathcal{X} is smooth is not easy to find in the literature, but it resembles the line of argument in Lerman's work on symplectic cuts [25].

Another way to regard this space \mathcal{X} is to use the Morse flow to induce maps from the quotient at each regular value to the quotients at the two adjacent critical values. This makes the symplectic quotients into an inverse system, and \mathcal{X} is the inverse limit.

In the body of the paper we were especially interested in an action of \mathbb{C}^{\times} on the Grassmannian $Gr_u(V \oplus W)$. Although we won't go into the details of the proof, it follows from

the well-known identification of symplectic quotients with Mumford quotients [27, §8.2] that in such a case \mathcal{X} is symplectomorphic to the inverse limit of Mumford quotients referred to in (5c) of the Main Theorem. The U(1) orbits of Morse flow lines get replaced by \mathbb{C}^{\times} -orbit closures, so equivalence classes of broken flows get replaced by nodal curves of genus 0: exactly those described in Proposition 7.3. The symplectic point of view probably will not imply any new results here. Nevertheless, thinking of points in the limit of Mumford quotients, or in the Chow quotient, as broken flows is a valuable aid to the intuition.

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